Towards a Mathematical Formulation of the Rodin Coil Torus

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Introduction

The following is an attempt to formalize the mathematics of the Rodin Torus. The goal is to attain a higher level of understanding of the Rodin Torus than can be obtained merely by observing the numerical sequences generating the Torus.

Key to the development is the use of decimal parity. Decimal parity is an operation that sums the digits in a number repeatedly to yield a single digit, the decimal parity digit for the original number.

For example the digits in the number 2,048 sum to 2+0+4+8 = 14, and the digits in 14 sum to 1+4 = 5. The decimal parity digit of 2048 is therefore 5.

It is interesting that all of the same results can be derived if the modulo operator is used in place of the decimal parity operator. The modulo operator is the remainder operator: x mod y is the remainder of x divided by y. The difference in the resulting patterns of digits is that everywhere there is a 9 decimal parity, there would be a 0 modulus. Since there is a one-to-one correspondence between the two approaches, the difference is apparently merely symbolic. Nonetheless, we shall use the decimal parity operator in this development, and leave the modulo development as an exercise for the bored reader with too much time on his or her hands.

In the development we discuss various series of numbers. Each such series has an *index*, which we start at 1 and number sequentially, one element at a time. (The series index could start at 0, but we are going to end up in matrices, which have an index starting at 1, so we'll start at 1 with our series.) The modulo operator is used for index arithmetic, since this is a more conventional approach. However, a purely decimal parity development is possible merely by substituting "decimal parity = 9" anywhere "modulus = 0" is used.

The Multiplicative Series

Let mi denote the infinite series with each element the decimal parity of the multiplication series for digit i,
i = 1, …, 9. E.G. for i = 2,

$$m2 = \{ 2, 4, 6, 8, 1, 3, 5, 7, 9, 2, 4, 6, 8, 1, 3, 5, \ldots\} \qquad [1]$$

Observation O1:

The series mi repeat with period 9. [O1]

Denote the j^{th} element of the series as ai_j, with j starting at 1. Observation 1 means

$$ai_j = ai_k \quad \text{iff} \quad j \bmod 9 = k \bmod 9 \qquad [2]$$

Now consider the pair of series m1 and m8.

$$m1 = \{ 1, 2, 3, 4, 5, 6, 7, 8, 9, 1, 2, \dots \} \quad [3]$$

$$m8 = \{ 8, 7, 6, 5, 4, 3, 2, 1, 9, 8, 7, \dots \} \quad [4]$$

Notice that

$$a1_9 = a8_9 = 9$$

$$a1_1 = a8_8$$

$$a1_2 = a8_7$$

$$a1_3 = a8_6$$

…and so on. We can state this more tersely (with the modulo operator taking precedence over the subtraction operator):

$$a1_{j \bmod 9} = a8_{(9-j \bmod 9)} \text{ when } j \bmod 9 \neq 0, \quad [5]$$

and

$$a1_j = a8_j = 9 \text{ when } j \bmod 9 = 0$$

Similarly,

$$a8_{j \bmod 9} = a1_{(9-j \bmod 9)} \text{ where } j \bmod 9 \neq 0 \quad [6]$$

The same observations of the series m4 and m5 lead to a similar conclusion:

$$m4 = \{ 4, 8, 3, 7, 2, 6, 1, 5, 9, 4, 8, \dots \} \quad [7]$$

$$m5 = \{ 5, 1, 6, 2, 7, 3, 8, 4, 9, 5, 1, \dots \} \quad [8]$$

$$a4_{j \bmod 9} = a5_{(9-j \bmod 9)} \text{ when } j \bmod 9 \neq 0 \quad [9]$$

$$a4_j = a5_j = 9 \text{ when } j \bmod 9 = 0$$

And

$$a5_{j \bmod 9} = a4_{(9-j \bmod 9)} \text{ where } j \bmod 9 \neq 0 \quad [10]$$

And finally the same observations of the series m2 and m7 lead to a similar conclusion:

$$m2 = \{ 2, 4, 6, 8, 1, 3, 5, 7, 9, 2, 4, \ldots \} \quad [11]$$

$$m7 = \{ 7, 5, 3, 1, 8, 6, 4, 2, 9, 7, 5, \ldots \} \quad [12]$$

$$a2_{j \bmod 9} = a7_{(9\text{-}j \bmod 9)} \text{ when } j \bmod 9 \neq 0 \quad [13]$$

$$a2_j = a7_j = 9 \text{ when } j \bmod 9 = 0$$

And

$$a7_{j \bmod 9} = a2_{(9\text{-}j \bmod 9)} \text{ where } j \bmod 9 \neq 0 \quad [14]$$

Next consider a different pattern in multiplication series, the m3 and m6 series.

$$m3 = \{ 3, 6, 9, 3, 6, 9, 3, 6, 9, 3, 6, 9, \ldots\} \quad [15]$$

$$m6 = \{ 6, 3, 9, 6, 3, 9, 6, 3, 9, 6, 3, 9, \ldots\} \quad [16]$$

This leads to the conclusions that, first, the series repeat,

$$a3_j = a3_{(j \bmod 3)} \quad [17]$$

$$a6_j = a6_{(j \bmod 3)} \quad [18]$$

and, second, that the series are related as follows:

$$a6_j = a3_{(3 - j \bmod 3)} \quad \text{iff} \quad j \bmod 3 \neq 0 \quad [19]$$

$$a6_j = a3_j \quad \text{iff} \quad j \bmod 3 = 0 \quad [20]$$

From these two series we can construct a new, artificial series, e, fabricated as follows:

$$e = \{a6_1, a3_2, a3_3, a3_4, a6_5, a6_6, a6_7, a3_8, a3_9, a3_{10}, \ldots\} \quad [21]$$

Or, numerically,

$$e = \{6, 6, 9, 3, 3, 9, 6, 6, 9, 3, 3, 9, 6, 6, 9, 3, 3, 9, \ldots \} \quad [22]$$

This series, which we call the *equivalence series*, has the representative term of

$$e = \{ \ldots, aXj, \ldots \} \quad [23]$$

where $X = 3$ if int((j+2)/3) odd
and $X = 6$ if even

Now consider the *doubling series*:

$$\{\ 2, 4, 6, 8, 16, 32, 64, 128, 256, 512, 1024, \ldots\ \} \qquad [24]$$

which has decimal parity of

$$d = \{\ 2, 4, 8, 7, 5, 1, 2, 4, 8, 7, 5, \ldots\ \} \qquad [25]$$

Observation:

This is a repeating series with period 6.

Or in other words, denoting the j^{th} element of this series by d_j,

$$d_j = d_k \qquad \text{iff} \qquad j \bmod 6 = k \bmod 6. \qquad [26]$$

Let the *reversed doubling series* be denoted by b:

$$b = \{\ 1, 5, 7, 8, 4, 2, 1, 5, 7, 8, 4, \ldots\ \} \qquad [27]$$

This also repeats with period 6. With the j^{th} element of b denoted by b_j,

$$b_j = d_{(7 - j \bmod 6)} \qquad \text{if} \qquad j \bmod 6 \neq 0 \qquad [28]$$

and

$$b_j = d_1 = 2 \qquad \text{if} \qquad j \bmod 6 = 0 \qquad [29]$$

The Torus

The torus is constructed from the above series.

Each element of the torus is an element of multiple series.

We will begin by considering the 2-dimensional surface of the torus. In two dimensions, each element of the torus is also an element of either the doubling circuit, the reverse doubling circuit, or the series e. Each element is also a member of two multiplicative series that are not pairs (in the sense that m1 and m8 are pairs.)

Let's first examine the 8154 torus. The surface of this torus contains the series m8, m1, m5, and m4.

Here is a fragment, with rows and columns numbered:

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**
1	6	6	9	3	3	9	6	6	9	3	3	9	6	6	9	3	3	9	6
2	5	1	2	4	8	7	5	1	2	4	8	7	5	1	2	4	8	7	5
3	2	1	5	7	8	4	2	1	5	7	8	4	2	1	5	7	8	4	2
4	9	6	6	9	3	3	9	6	6	9	3	3	9	6	6	9	3	3	9
5	7	5	1	2	4	8	7	5	1	2	4	8	7	5	1	2	4	8	7
6	4	2	1	5	7	8	4	2	1	5	7	8	4	2	1	5	7	8	4
7	3	9	6	6	9	3	3	9	6	6	9	3	3	9	6	6	9	3	3
8	8	7	5	1	2	4	8	7	5	1	2	4	8	7	5	1	2	4	8
9	8	4	2	1	5	7	8	4	2	1	5	7	8	4	2	1	5	7	8
10	3	3	9	6	6	9	3	3	9	6	6	9	3	3	9	6	6	9	3
11	4	8	7	5	1	2	4	8	7	5	1	2	4	8	7	5	1	2	4
12	7	8	4	2	1	5	7	8	4	2	1	5	7	8	4	2	1	5	7
13	9	3	3	9	6	6	9	3	3	9	6	6	9	3	3	9	6	6	9
14	2	4	8	7	5	1	2	4	8	7	5	1	2	4	8	7	5	1	2
15	5	7	8	4	2	1	5	7	8	4	2	1	5	7	8	4	2	1	5
16	6	9	3	3	9	6	6	9	3	3	9	6	6	9	3	3	9	6	6
17	1	2	4	8	7	5	1	2	4	8	7	5	1	2	4	8	7	5	1
18	1	5	7	8	4	2	1	5	7	8	4	2	1	5	7	8	4	2	1
19	6	6	9	3	3	9	6	6	9	3	3	9	6	6	9	3	3	9	6

Imagine the surface of the torus as a matrix, starting at the element $t_{1\ 1}$, which is in the upper left corner: a 6. The first subscript is the row, and the second is the column.

For the 8154 torus, the following conditions hold:

$$t_{1\ x} = e \qquad [30]$$

where $t_{1\ x}$ refers to the first row of the matrix.

(Taking e_1 as the first element of the matrix is arbitrary. We could have taken any element in e, d, or b as the first element, and still have been able to construct the following formulae. You can see this is so because e_1, d_1, and b_1 all appear in the first column in some row (look at rows 14 and 18 for d and b.) In fact there is no reason to start with the first element of either of these three series, since there is a row starting with each element of each of the series, and any row could be the first row. If a different origin were chosen, certain constants in the following development would be different, but the results would otherwise be the same. These are some of the constants which are added to indices to make them match the matrix pattern. Therefore do not focus overly on constants used as addends in index arithmetic. Many of the multiplicative constants, on the other hand, are structural and would not change.)

Also,

$$t_{2\,1} = d_5 \qquad [31]$$

and in general,

$$t_{2\,j} = d_{j+4} \qquad [32]$$

Similarly:

$$t_{3\,j} = b_{j+5} \qquad [33]$$

Further examination of the 8154 torus shows that

$$t_{4\,j} = e_{j+5} \qquad [34]$$

Notice that the next element of the 4th row $t_{4\,2} = e_1$, and $t_{5\,2} = d_5$, and $t_{6\,2} = b_6$. In other word $t_{4\,2} = t_{1\,1}$, $t_{5\,2} = t_{2\,1}$, and $t_{6\,2} = t_{3\,1}$. The second set of three rows is the same as the first set, shifted one column to the right. This shift is the reason why the matrix we are examining lies on the surface of a torus. Continuing,

$$t_{5\,j} = d_{j+3}$$

$$t_{6\,j} = b_{j+4}$$

$$t_{7\,j} = e_{j+4}$$

$$t_{8\,j} = d_{j+2}$$

$$t_{9\,j} = b_{j+3}$$

$$t_{10\,j} = e_{j+3}$$

$$t_{11\,j} = d_{j+1}$$

$$t_{12\,j} = b_{j+2}$$

$$t_{13\,j} = e_{j+2}$$

$$t_{14\,j} = d_j$$

$$t_{15\,j} = b_{j+1}$$

$$t_{16\,j} = e_{j+1}$$

$t_{17\,j} = d_{j+5}$

$t_{18\,j} = b_j$

After row 18, the rows repeat: $t_{19\,x} = t_{1\,x}$, $t_{20\,x} = t_{2\,x}$, so that in general

$$t_{j\,x} = t_{(j \bmod 18)\,x} \qquad [35]$$

Also, after 18 columns, the columns repeat:

$$t_{x\,k} = t_{x\,(k \bmod 18)} \qquad [36]$$

so that

$$t_{j\,k} = t_{(j \bmod 18)\,(k \bmod 18)} \qquad [37]$$

We now take the rather unconventional step (from the viewpoint of matrix algebra) of reading across the rows and columns diagonally. For this to work we need to establish an equivalent to a left-to-right direction. We arbitrarily designate up-and-right as left-to-right, and up-and-left as left-to-right. This tells us which direction in which to number our series as they increase. (This convention can be reversed without loss of results, but the m8 series would become the m1 series, and vice-versa, and the m5 series would become the m4 series, and vice-versa. This follows from the fact that they are the reverse of each other, so reversing the direction convention would exchange the series.)

Using this convention, we note the following:

$$m8_1 = t_{9\,1} \qquad [38]$$

$m8_2 = t_{8\,2}$

$m8_3 = t_{73}$

and in general

$$m8_j = t_{(10-j)\,j} \quad \text{while } j < 10 \qquad [39]$$

Furthermore, this diagonal series repeats with period 6:

$$m8_1 = t_{9\,7},\ m8_2 = t_{8\,8} \qquad [40]$$

Even more interesting is that the m8 series lies in every other diagonal row. For example,

$$t_{9\,1} = m8_1 \qquad [41]$$

$t_{9\,3} = m8_7$

$t_{9\,5} = m8_4$

and so on.

On the next row,

$t_{10\,2} = m8_6$ [42]

$t_{10\,4} = m8_3$

$t_{10\,6} = m8_9$

and on row 11:

$t_{11\,1} = m8_5$ [43]

$t_{11\,3} = m8_2$

$t_{11\,5} = m8_8$

These triples then repeat. (The m8 indices in these series are called the *family number groups* by Marko Rodin.)

A similar pattern exists for m1, where $t_{17\,1} = m1_1$, and so forth. Therefore each element of each row is also an element of either m1 or m8. If the first element of the row is a member of m1, the next element is a member of m8, and vice-versa.

Denote by "dep(n)" the function of taking the decimal parity of the number n, as defined in the Introduction. Notice the general form for a row x starting with element $m8_h$ followed by element $m1_i$

$t_{x\,k} = m8_{dep(h+3(k-1))}$ for k odd [44]

$t_{x\,k} = m1_{dep(i+3(k-2))}$ for k even [45]

Similarly, if a row x starts with $m1_h$ followed by $m8_i$, then

$t_{x\,k} = m1_{dep(h+3(k-1))}$ for k odd [46]

$t_{x\,k} = m8_{dep(i+3(k-2))}$ for k even [47]

Observe next the first element of the 8154 torus. This must be an element of m8 because $t_{2\,1}$ and $t_{1\,2}$ are the $m1_5$ and $m1_6$ elements of m1, so $t_{1\,1}$ must be in a diagonal containing the series m8. Therefore, since $t_{1\,1}$ is a 6,

$t_{1\,1} = m8_3$ $t_{1\,2} = m1_6$ [48]

We also observe the following pattern along the rows of the torus:

$t_{2\,1} = m1_5$ $t_{2\,2} = m8_8$ [49]

$t_{3\,1} = m8_7$ $t_{3\,2} = m1_1$

$t_{4\,1} = m1_9$ $\quad$ $t_{4\,2} = m8_3$

$t_{5\,1} = m8_2$ $\quad$ $t_{5\,2} = m1_5$

$t_{6\,1} = m1_4$ $\quad$ $t_{6\,2} = m8_7$

$t_{7\,1} = m8_6$ $\quad$ $t_{7\,2} = m1_9$

$t_{8\,1} = m1_8$ $\quad$ $t_{8\,2} = m8_2$

$t_{9\,1} = m8_1$ $\quad$ $t_{9\,2} = m1_4$

$t_{10\,1} = m1_3$

$t_{11\,1} = m8_5$

$t_{12\,1} = m1_7$

$t_{13\,1} = m8_9$

$t_{14\,1} = m1_2$

$t_{15\,1} = m8_4$

$t_{16\,1} = m1_6$

$t_{17\,1} = m8_8$

$t_{18\,1} = m1_1$

In general we see that

$t_{j\,1} = m8_{dep(2j+1)}$ and $t_{j\,2} = m1_{dep(2(j+2))}$ when j is odd [50]

$t_{j\,1} = m1_{dep(2j+1)}$ and $t_{j\,2} = m1_{dep(2(j+2))}$ when j is even [51]

From [44-47] and [50-51] we can see the general term of the 8154 torus for any element $t_{j\,k}$ is

$t_{j\,k} = m8_{dep(dep(2j+1)+3(k-1))}$ for j odd and k odd [52]

$t_{j\,k} = m1_{dep(dep(2(j+2))+3(k-2))}$ for j odd and k even [53]

$t_{j\,k} = m1_{dep(dep(2j+1)+3(k-1))}$ for j even and k odd [54]

$t_{j\,k} = m8_{dep(dep(2(j+2))+3(k-2))}$ for j even and for k even [55]

At this point we see that any element $t_{j\,k}$ is determined both by [30-34] and also by [52-55]. Next we will show that the same element is also determined by m4 and m5 in similar fashion.

It is not difficult to see that the diagonal to the upper left is m5, with

$t_{1\,1} = m5_3$

Let's look at the m5 series lying in every other row element. For example,

$t_{9\,1} = m5_7$ [58]

$t_{9\,3} = m5_4$

$t_{9\,5} = m5_1$

which then repeats as the row continues. Likewise on the next row,

$t_{10\,2} = m5_6$ [59]

$t_{10\,4} = m5_3$

$t_{10\,6} = m5_9$

and on row 11:

$t_{11\,1} = m5_8$ [60]

$t_{11\,3} = m5_5$

$t_{11\,5} = m5_2$

Once again we see the family number groups in the indices here.

If a row x starts with $m5_h$ and is followed by $m4_i$, the general expression for the k^{th} element is

$t_{x\,k} = m5_{dep(h+3(k-1))}$ for k odd [58]

$t_{x\,k} = m4_{dep(i+3(k-2))}$ for k even [59]

Similarly, if a row x starts with $m4_h$ followed by $m5_i$, then

$t_{x\,k} = m4_{dep(h+3(k-1))}$ for k odd [60]

$t_{x\,k} = m5_{dep(i+3(k-2))}$ for k even [61]

These are identical in form to equations [44-47].

Now let's continue as before, taking a look at how h and i are determined for a row starting with $m5_h$ followed by $m4_i$, or vice-versa.

$t_{1\,1} = m5_3 \qquad t_{1\,2} = m4_6$ [62]

We also observe the following pattern along the rows of the torus:

$t_{2\,1} = m4_8 \qquad t_{2\,2} = m5_2$ [63]

$t_{3\,1} = m5_4 \qquad t_{3\,2} = m4_7$

$t_{4\,1} = m4_9 \qquad t_{4\,2} = m5_3$

$t_{5\,1} = m5_5 \qquad t_{5\,2} = m4_8$

$t_{6\,1} = m4_1 \qquad t_{6\,2} = m5_4$

$t_{7\,1} = m5_6 \qquad t_{7\,2} = m4_9$

$t_{8\,1} = m4_2 \qquad t_{8\,2} = m5_5$

$t_{9\,1} = m5_7 \qquad t_{9\,2} = m4_1$

$t_{10\,1} = m4_3 \qquad \ldots$

$t_{11\,1} = m5_8$

$t_{12\,1} = m4_4$

$t_{13\,1} = m5_9$

$t_{14\,1} = m4_5$

$t_{15\,1} = m5_1$

$t_{16\,1} = m4_6$

$t_{17\,1} = m5_2$

$t_{18\,1} = m4_7$

In general we see that

$t_{j\,1} = m5_{dep(3+(j-1)/2)}$ and $t_{j\,2} = m4_{dep(6+(j-1)/2)}$ when j is odd [64]

$t_{j\,1} = m4_{dep(8+(j-2)/2)}$ and $t_{j\,2} = m5_{dep(2+(j-2)/2)}$ when j is even [65]

Equations [64-65] are in a form where it is pretty easy to see where they came from, by looking at the patterns in [63], but they are long-winded, and can be simplified to the equivalent

$t_{j\,1} = m5_{dep((j+5)/2)}$ and $t_{j\,2} = m4_{dep((j+11)/2)}$ when j is odd [66]

$t_{j\,1} = m4_{dep((j+14)/2)}$ and $t_{j\,2} = m5_{dep((j+2)/2)}$ when j is even [67]

From [58-61] and [64-65] we can see the general term of the 8154 torus for any element $t_{j\,k}$ is

$t_{j\,k} = m5_{dep(dep((j+5)/2)+3(k-1))}$ for j odd and k odd [68]

$t_{j\,k} = m4_{dep(dep((j+11)/2)+3(k-2))}$ for j odd and k even [69]

$t_{j\,k} = m4_{dep(dep((j+14)/2)+3(k-1))}$ for j even and k odd [70]

$t_{j\,k} = m5_{dep(dep((j+2)/2)+3(k-2))}$ for j even and k even [71]

We see now that [52-55] and [68-71] describe the same elements of the torus, the first set of equations using m8 and m1, the second set using m5 and m4. This is in addition to the same elements being described by the doubling, reverse doubling, and equivalence series as shown in [30-37]. Each element is therefore triply determined.

Enumeration of the Rodin Tori

We have discussed the 8154 Rodin Torus. Is it the only torus surface which can be created so that each point is multiply determined? Simply put the answer is, "No."

Consider the torus constructed as follows:

e_2 [72]

d_6

b_1

The first few elements of this torus look like:

e2	*6*	*9*
d6	1	2
b1	1	5
e1	*6*	*6*

In this picture we show the first two elements of e_2, d_6, and b_1, followed by e_1 which forms the next row. (Our primitive tools do not permit us to use subscripting for indexes in the pictures: sorry. Please use your imagination.) Refer to Appendix A for the d, b, and e series. This is the 1845 torus, since m1 passes diagonally to the upper right through the e_2's 6, and m4 passes diagonally to the upper left:

e2	*6*	*9*
d6	1	2
b1	1	5
e1	*6*	*6*

Note that it is redundant to call this 1845, since if m1 is passing diagonally up and right through $t_{1\ 1}$ then m8 must be parallel through $t_{2\ 2}$. Similarly if m4 is passing diagonally up and left through $t_{1\ 1}$, then m5 must be passing parallel through $t_{2\ 1}$. Since m8 is implied by the existence of m1, and m5 is implied by the existence of m4, we can call this the 14 torus and say just as much as if we called it the 1845 torus.

The observant reader will have noticed that the above torus is in fact only rows 16 through 19 of the 8154 discussed in the previous section torus (aka 85 torus by in our new, abbreviated nomenclature.) The only difference is in the choice of origin. In fact we do not really think of these as being separate tori at all, since they differ only in point of origin, and after all we did choose to start with e_1 arbitrarily. So the 14 torus is equivalent—if not identical—to the 85 torus.

Thus far we really have only one torus. Are there others that are truly different? The simple answer is "Yes."

Let's start by referring to Appendix A, which shows the doubling, backwards, and equivalence series for reference. If a torus were to have any two rows one after the other with both starting with d_1, it would look like

2	4
2	4

By referring to Appendix B, you can easily verify that there is no m-series with the sequence …4,2,…. Therefore this does not define a Rodin torus .

The same can be said for the rows starting with d_1 followed by d_2, and so on. This leads to the conclusion that the d row must not be followed by a d row for a Rodin torus to emerge.

A similar set of observations leads to the conclusion that a b row must not be followed by another b row.

Even if a d row is followed by a b row, a Rodin torus is not always created. For example if a d_1 row is followed by a b_1 row, the result is not a Rodin torus:

2	4
1	5

There is no m-series with 1 followed by 4, or with 5 followed by 2 (see Appendix B again.

In Appendix C we have listed exhaustively the rows starting with d_j, 0<j<7, and then following with each possible row b_k with 0<k<7. These entries look like:

			2			
		e6	*9*	*6*		
	1		4	8		
			1	5		
		e5	*3*	*9*	**28**	

The red (if you have a color copy) outer numbers in bold indicate the indices for d (on the top) and b (to the left.) In this case we have d_2 and b_1. The intersection of the column for the d index and the row for the b index is the origin of the matrix in each case. (We abandon at this point the notion that the origin must be e_1. Since it is arbitrary we can set it where we like.) This d_1 element is below the e6 line in this example, and contains a 4 as d_2. The next cell to the right is d_3 = 8. Below are the first two elements of b_1: 1, 5 (see Appendix A.)

These 4 cells define the torus: reading from b_1 up and to the right we see the m-series 1,8, which is m7. Since this diagonal is one diagonal below the origin, we know the diagonal up and right through the origin must m2. Up and left through the origin is the series 5,4, which is m8. This is therefore the 28 Rodin torus.

Knowing this is the 28 torus permits us to deduce the e rows above and below the d-b row pair. For example we know the up-right diagonal through the 5 cell must be an m2, and 5 is preceded by 3 in m2, so below the 1 we can wrote a 5. Similarly we can fill in the other e series slots, and deduce that e5 is following b1, while e6 is preceding d_2. The e-series are a result of this being a 28 torus; it is not hard to see that nothing else will work.

Appendix C therefore contains an exhaustive list of Rodin tori which can be constructed from rows in which a d row is followed by a b row.

Similarly Appendix D contains an exhaustive list of Rodin tori which can be constructed from rows in which a b row is followed by a d row.

Now the m-series that map onto the Rodin torus are the m1 & m8, m4 & m5, and m2 & m7 pairs. It is not hard to show that the two m-series passing through the origin in a Rodin torus cannot be pairs. In other words an 88 or an 81 torus is not possible.

You have only to try it to see it: here is an 88 torus:

7		7
	8	
6		6

If the 8 is in the origin, you see that we would have to have a d or b row with 7, x, 7, where x is any d or b series number. But no such sequence exists. (Similarly 6, x, 6 is not an e series.) So an 88 torus cannot be built. Similar trials show that a torus must have components from two separate number pairs.

Therefore, although there are 6 m-series, there are not 36 possible Rodin tori. Here is a table, with blank entries for those we know cannot be built.

	1	***2***	***4***	***5***	***7***	***8***
1		12	14	15	17	
2	21		24	25		28
4	41	42			47	48
5	51	52			57	58
7	71		74	75		78
8		82	84	85	87	

So ther are 24 possible Rodin tori, at least from this point of view. But we have shown that restrictions on the placement of rows, such as adjacent d and b rows, prevents the formation of all possibilities. In fact only 6 Rodin tori can actually be constructed, as shown in Appendices C and D.

It may appear to you that there are actually 12 tori in the Appendices. Remember that because of pairing of series, some tori which look different at the origin are actually identical: 85 torus = 14 torus, for example. Here is the above table, with the possible tori only in large font, and the impossible ones smaller:

	1	***2***	***4***	***5***	***7***	***8***
1		12	14	15	17	
2	21		24	25		28
4	41	42			47	48
5	51	52			57	58
7	71		74	75		78
8		82	84	85	87	

Here are the possible equivalent Rodin tori:

14	85
17	82
25	74
28	71
41	58
47	52

We will use either of these pair members to denote them both interchangeably.

<u>The 3D Rodin Torus</u>

Now that we know how many different 2D tori can be constructed, it is tempting to try to construct a 3D torus.

Consider the 85 torus we discussed first. We can represent this as a vector of series going down the page. Above we showed this as extending off to the right:

$$\begin{array}{llll} e_1 & e_2 & e_3 & \dots \\ d_5 & d_6 & d_1 & \dots \\ b_6 \dots \\ e_6 \dots \\ d_4 \dots \\ b_5 \dots \\ e_5 \dots \\ \dots \end{array} \quad [73]$$

Suppose instead we look at this series from the left edge:

$$\begin{array}{llll} e_1 & e_2 & e_3 & \dots \\ d_5 & d_6 & d_1 & \dots \\ b_6 \dots \\ e_6 \dots \\ d_4 \dots \\ b_5 \dots \\ e_5 \dots \\ \dots \end{array} \quad [74]$$

We see the starting element of each row, but the other elements extend down into the paper and are hidden from view. This is no disadvantage, however, since we know from the starting element all the elements that must follow in the series:

e_1 [75]
d_5
b_6
e_6
d_4
b_5
…

Now lets try to build the same series off to the right, remembering that we see only the first element of each row: each row will extend down into the paper:

e_1	d_5	b_6	e_6	d_4	b_5	e_5	…	[76]
d_5								
b_6								
e_6								
d_4								
b_5								

We now have two intersecting tori; they intersect at the e_1 series in the corner. To really have a 3D torus, we need to fill in the blanks.

According to Appendix C, d_5 can be followed by either of e_1, e_3, or e_5. Let's plunge in and choose e_1 arbitrarily. (We will see in a moment that this choice is not crucial.)

e_1	d_5	b_6	e_6	d_4	b_5	e_5	…	**14**	**[77]**
d_5	$\mathbf{e_1}$	**?**							
b_6									
e_6									
d_4									
b_5									
…									
14									

The number in bold is the torus number, found using the first complete d-b or b-d pair in the row or column, then looking it up in Appendix C or D, respectively.

Now notice the ?: b_6 must be followed by a d in its column. But notice also that the e_1 we just added must be followed by a b in its row, since it is preceded by a d. So in the spot marked with a **?**, there is no row that can work.

Hence we cannot build a 3D torus if <u>*both*</u> *original intersecting tori are in the d, b, e sequence.*

We speculate that the same will hold true if both are in the b, d, e sequence.

Let's therefore try to build one by adding a b, d, e sequence to the right instead. We'll choose the sequence to the right as a 25 (aka 74) torus.

It is useful to notice from Appendices C and D that d, b, e sequence indices always decrease while the b, d, e indices always increase. In a d, b, e sequence, if we have d_i , b_j, e_k, then next we'll see d_{i-1}, b_{j-1}, e_{k-1} (unless i, j, or k =1, in which case we'll see a 6 next. Similarly if we have b_i, d_j, e_k we'll see next b_{i+1}, d_{j+1}, and e_{k+1}, (unless i, j, or k = 6, in which case we'll see a 1 next.) These observations help us construct the tori as we proceed.

Choose again e_1 for the first blank position:

e_1	b_5	d_2	e_2	b_6	d_3	…	**25**	[78]
d_5	$\mathbf{e_1}$	?						
b_6								
e_6								
d_4								
b_5								
…								
14								

From Appendix C possible followers of d_2 for the ? spot are b_1, b_3, or b_5. From Appendix D each of these may have a predecessor of e_1 on the second row. Choose b_1; this determines the rest of the row to the right:

e_1	b_5	d_2	e_2	b_6	d_3	…	**25**	[79]
d_5	e_1	$\mathbf{b}_1$	d_6	e_2	b_2	…	**58**	
b_6	?							
e_6								
d_4								
b_5								
…								
14								

There are no more choices: the tori are now completely determined.

For example the spot where the ? rests now is also determined. b_5 followed by e_1 must be (from Appendix C) a 52 (aka 47) torus. The question mark must therefore be d_1 followed downwards by b_4; the remainder of this column is now determined:

e_1	b_5	d_2	e_2	b_6	d_3	…	**25**	[80]
d_5	e_1	b_1	d_6	e_2	b_2	…	**58**	
b_6	$\mathbf{d}_1$							
e_6	b_4							
d_4	e_6							
b_5	d_6							
…	…							
14	**47**							

On the third row b_6, d_1 defines a 17 torus, so we get:

e_1	b_5	d_2	e_2	b_6	d_3	…	**25**	[81]
d_5	e_1	b_1	d_4	e_6	b_6	…	**58**	
b_6	d_1	$\mathbf{e}_5$	b_1	d_2	e_6	…	**17**	
e_6	b_4							
d_4	e_6							
b_5	d_6							
…	…							
14	**47**							

Filling out the remainder of the grid we get:

e_1	b_5	d_2	e_2	b_6	d_3	…	**25**	[82]
d_5	e_1	b_1	d_6	e_2	b_2	…	**58**	
b_6	d_1	e_5	b_1	d_2	e_6	…	**17**	
e_6	b_4	d_1	e_1	b_5	d_2	…	**74**	
d_4	e_6	b_6	d_5	e_1	b_1	…	**41**	
b_5	d_6	e_4	b_6	d_1	e_5	…	**82**	
…	…	…	…	…	…			
14	**47**	**28**	**85**	**52**	**71**			

Each of the bold numbers is labeling an infinite plane extending down from the surface of the paper, each holding the surface of a Rodin torus. This means that each point is determined by **4** multiplicative series as well as 2 of the d, b, or e series of which it is an element. *Thus each element is locked into place by being a member of no less than 6 series at once.* This is no small amount of regularity!

Notice also that we have used all twelve of the permissible Rodin tori so far. Let's go one more in each direction and see what happens:

e_1	b_5	d_2	e_2	b_6	d_3	e_3	…	**25**	[83]
d_5	e_1	b_1	d_6	e_2	b_2	d_1	…	**58**	
b_6	d_1	e_5	b_1	d_2	e_6	b_2	…	**17**	
e_6	b_4	d_1	e_1	b_5	d_2	e_2	…	**74**	
d_4	e_6	b_6	d_5	e_1	b_1	d_6	…	**41**	
b_5	d_6	e_4	b_6	d_1	e_5	b_1	…	**82**	
e_5	b_3	d_6	e_6	b_4	d_1	e_1	…	**<u>25</u>**	
…	…	…	…	…	…	…			
14	**47**	**28**	**85**	**52**	**71**	**<u>14</u>**			

At this point we can't be too surprised that the series of tori looks like it is going to repeat.

It is worth pointing out that the columns are filled with d, b, e series, while the rows are filled with b, d, e series.

Having constructed a 3D Rodin torus, it is worth asking whether there is more than one. This should be our next issue.
Let's upgrade our torus notation to 3 dimensions. $t_{i\,j\,k}$ is now the torus element, with i denoting the index of the row down the page, j denoting the index of the column across the page, and k denoting the index of the element extending perpendicular to the surface of the page.

Recall in [78] that after choosing $t_{1\,2\,1} = e_1$, we had three choices for the ?
($t_{2\,2\,1}$): b_1, b_3, and b_5. We chose b_1 and found this determined the torus of [82] (no pun here with the 82 torus.)

Let's try b_3 instead of b1:

e_1	b_5	d_2	e_2	b_6	d_3	…	**25**	[84]
d_5	e_1	$\mathbf{b_3}$	d_6	…			**X**	
b_6								
e_6								
d_4								
b_5								
…								
14								

If $t_{2\,3\,1} = b_3$, then $t_{2\,4\,1}$ must be d_6, because $t_{2\,1\,1} = d_5$, and since this is a b-d-e sequence, the next d index must be 5+1 = 6. But Appendix D says that in the 25 torus determined by b_3, d_6, the preceding row must be e_5, not e_1 as in [78]. Therefore b_3 cannot be a candidate for $t_{2\,3\,1}$.

Similarly the predecessor of b_5, d_6 must be e_3, so b_5 $t_{2\,3\,1}$. $t_{2\,3\,1} = b_1$ is the only candidate that produces a 3D Rodin Torus.

What about using a different choice for $t_{2\,2\,1}$. Previously we tried e_1, and that worked. But recall that e_3 and e_5 were legal candidates. We can see that these should work, just by the logic of the preceding two paragraphs. Let's try $t_{2\,2\,1} = e_3$:

e_1	b_5	d_2	e_2	b_6	d_3	…	**25**	[85]
d_5	$\mathbf{e_3}$	b_5	d_6	e_4	b_6	…	**82**	
b_6	d_5	e_1	b_1	d_6	e_2	…	**41**	
e_6	b_4	d_1	e_1	b_5	e_2	…	**74**	
d_4	e_2	b_4	d_5	e_3	b_5	…	**17**	
b_5	d_4	e_6	b_6	d_5	e_1	…	**58**	
…	…	…	…	…	…			
14	**71**	**52**	**85**	**28**	**47**			

This is our second 3D Rodin torus. Notice that the m-series making up this 3D torus are the same set of 12 m-series making up [82], but in the reverse order.

We must of course try e_5 next:

e_1	b_5	d_2	e_2	b_6	d_3	…	**25**	[86]
d_5	$\mathbf{e_5}$	b_3	d_6	e_6	b_4	…	**25**	
b_6	d_3	e_3	b_1	d_4	e_4	…	**74**	
e_6	b_4	d_1	e_1	b_5	d_2	…	**74**	
d_4	e_4	b_2	d_5	e_5	b_3	…	**74**	
b_5	d_2	e_2	b_6	d_3	e_3	…	**25**	
…	…	…	…	…	…			
14	**14**	**85**	**85**	**85**	**14**			

Here only 4 of the 12 possible m-series are used to build the torus, and since they are pairs, there are really only 2 in use: 25 and 14.

What's Next

From this point there are several research directions of interest. One is to understand in a precise way how the number series lay on the surface of the torus. Another is to catalog the complete set of 3D tori, much as was done for the 2D tori in Appendices C and D. It is also interesting to conjecture that a 4D or higher dimensional torus might exist.

In the long run there are a number of fields of mathematics which are—with this work—now potentially applicable to the Rodin torus. These include matrix algebra, vector calculus, topology, and time series analysis. These in turn render much of physics accessible, including in particular classical electrodynamics.

Appendix A: d, b, and e Series

	1	**2**	**3**	**4**	**5**	**6**
d	2	4	8	7	5	1
	1	**2**	**3**	**4**	**5**	**6**
b	1	5	7	8	4	2
	1	**2**	**3**	**4**	**5**	**6**
e	6	6	9	3	3	9

Appendix B: M-Series

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
m1	1	2	3	4	5	6	7	8	9
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
m8	8	7	6	5	4	3	2	1	9
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
m4	4	8	3	7	2	6	1	5	9

	1	2	3	4	5	6	7	8	9
m5	5	1	6	2	7	3	8	4	9
	1	2	3	4	5	6	7	8	9
m2	2	4	6	8	1	3	5	7	9
	1	2	3	4	5	6	7	8	9
m7	7	5	3	1	8	6	4	2	9

Appendix C: All Possible d-b Rodin Tori

b: \ d:	1	2	3	4	5	6
1	2 4 1 5 **X**	*e6* 9 6 4 8 1 5 *e5* 3 9 **28**	8 7 1 5 **X**	*e4* 3 3 7 5 1 5 *e3* 9 3 **52**	5 1 1 5 **X**	*e2* 6 9 1 2 1 5 *e1* 6 6 **85**
2	*e3* 9 3 2 4 5 7 *e2* 6 9 **14**	4 8 5 7 **X**	*e1* 6 6 8 7 5 7 *e6* 9 6 **71**	7 5 5 7 **X**	*e5* 3 9 5 1 5 7 *e4* 3 3 **47**	1 2 5 7 **X**
3	2 4 7 8 **X**	*e4* 3 3 4 8 7 8 *e3* 9 3 **85**	8 7 7 8 **X**	*e2* 6 9 7 5 7 8 *e1* 6 6 **28**	5 1 7 8 **X**	*e6* 9 6 1 2 7 8 *e5* 3 9 **52**
4	*e1* 6 6 2 4 8 4 *e6* 9 6 **47**	4 8 8 4 **X**	*e5* 3 9 8 7 8 4 *e4* 3 3 **14**	7 5 8 4 **X**	*e3* 9 3 5 1 8 4 *e2* 6 9 **71**	1 2 8 4 **X**
5	2 4 4 2 **X**	*e2* 6 9 4 8 4 2 *e1* 6 6 **52**	8 7 4 2 **X**	*e6* 9 6 7 5 4 2 *e5* 3 9 **85**	5 1 4 2 **X**	*e4* 3 3 1 2 4 2 *e3* 9 3 **28**
6	*e5* 3 9 2 4 2 1 *e4* 3 3 **71**	4 8 2 1 **X**	*e3* 9 3 8 7 2 1 *e2* 6 9 **47**	7 5 2 1 **X**	*e1* 6 6 5 1 2 1 *e6* 9 6 **14**	1 2 2 1 **X**

Appendix D: All Possible b-d Rodin Tori

d: \ b:	1	2	3	4	5	6
1	1 5 2 4 **X**	*e2* *6 9* 5 7 2 4 *e3* *9 3* **41**	7 8 2 4 **X**	*e6* *9 6* 8 4 2 4 *e1* *6 6* **74**	4 2 2 4 **X**	*e4* *3 3* 2 1 2 4 *e5* *3 9* **17**
2	*e5* *3 9* 1 5 4 8 *e6* *9 6* **82**	5 7 4 8 **X**	*e3* *9 3* 7 8 4 8 *e4* *3 3* **58**	8 4 4 8 **X**	*e1* *6 6* 4 2 4 8 *e2* *6 9* **25**	2 1 4 8 **X**
3	1 5 8 7 **X**	*e6* *9 6* 5 7 8 7 *e1* *6 6* **17**	7 8 8 7 **X**	*e4* *3 3* 8 4 8 7 *e5* *3 9* **41**	4 2 8 7 **X**	*e2* *6 9* 2 1 8 7 *e3* *9 3* **74**
4	*e3* *9 3* 1 5 7 5 *e4* *3 3* **25**	5 7 7 5 **X**	*e1* *6 6* 7 8 7 5 *e2* *6 9* **82**	8 4 7 5 **X**	*e5* *3 9* 4 2 7 5 *e6* *9 6* **58**	2 1 7 5 **X**
5	1 5 5 1 **X**	*e4* *3 3* 5 7 5 1 *e5* *3 9* **74**	7 8 5 1 **X**	*e2* *6 9* 8 4 5 1 *e3* *9 3* **17**	4 2 5 1 **X**	*e6* *9 6* 2 1 5 1 *e1* *6 6* **41**
6	*e1* *6 6* 1 5 1 2 *e2* *6 9* **58**	5 7 1 2 **X**	*e5* *3 9* 7 8 1 2 *e6* *9 6* **25**	8 4 1 2 **X**	*e3* *9 3* 4 2 1 2 *e4* *3 3* **82**	2 1 1 2 **X**

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